W3L5 - POWER SERIES SOLUTIONS OF DIFFERENTIAL EQUATIONS

Q: Why are they needed?
A: Many Diffeq's can't be solved explicitly in terms of finite combinations of simple familiar functions.

Solve $y' - y = 0$

Lets assume a solution of the form:

$$y = f(x) = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots$$

$$= \sum_{n=0}^{\infty} c_n x^n \text{, exists}$$

$$y' = \sum_{n=1}^{\infty} n \cdot c_n x^{n-1}$$

$$\sum_{n=1}^{\infty} n \cdot c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1) c_{n+1} x^n - c_n x^n \right] = 0$$

$$\sum_{n=0}^{\infty} \underbrace{\left[(n+1) c_{n+1} - c_n \right]}_{\text{has to } = 0} x^n = 0$$

$$\Rightarrow (n+1) c_{n+1} - c_n = 0$$

$$c_{n+1} = \frac{c_n}{n+1}$$

$$c_{n+1} = \frac{c_n}{n+1} \qquad n = 0$$

$n=0 \quad c_1 = \frac{c_0}{1} \Rightarrow c_1 = c_0$

$n=1 \quad c_2 = \frac{c_1}{2} \Rightarrow c_2 = \frac{c_0}{2}$

$n=2 \quad c_3 = \frac{c_2}{3} \Rightarrow \frac{1}{3} \cdot \frac{c_0}{2} \Rightarrow \frac{c_0}{3!}$

$n=3 \quad c_4 = \frac{c_3}{4} \Rightarrow \frac{1}{4} c_3 \Rightarrow \frac{c_0}{4!}$

$$\underline{c_n = \frac{c_0}{n!}}$$

$$y' - y = 0$$

$$y = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{c_0}{n!} x^n = c_0 \underbrace{\sum_{n=0}^{\infty} \frac{x^n}{n!}}_{e^x}$$

$$\boxed{y = c_0 e^x}$$

EX 2: $y'' + y = 0$

Let $y = \sum_{n=0}^{\infty} c_n x^n$

$y' = \sum_{n=1}^{\infty} c_n \cdot n \cdot x^{n-1}$

$y'' = \sum_{n=2}^{\infty} c_n \cdot n \cdot (n-1) x^{n-2}$

$$\sum_{n=2}^{\infty} c_n (n)(n-1) x^{n-2} + \sum_{n=0}^{\infty} c_n x^n = 0$$

subtract add

to change the starting n of a summation

$$\sum_{n=0}^{\infty} \left[c_{n+2} (n+2)(n+1) x^n + c_n x^n \right] = 0$$

$$\sum_{n=0}^{\infty} \left[\underbrace{c_{n+2} (n+2)(n+1) + c_n}_{\text{has to } = 0 \text{ to get a solution}} \right] x^n = 0$$

assumption: $x \neq 0$

$$c_{n+2} (n+2)(n+1) + c_n = 0$$

$$c_{n+2} = \frac{-c_n}{(n+2)(n+1)}$$

$n=0$ $\quad c_2 = \frac{-c_0}{2 \cdot 1}$

$n=1$ $\quad c_3 = \frac{-c_1}{3 \cdot 2}$

$n=2$ $\quad c_4 = \frac{-c_2}{4 \cdot 3}$

$$= \frac{-(-c_0)}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{c_0}{4!}$$

$n=3$ $\quad c_5 = \frac{-c_3}{5 \cdot 4}$ ← sub

$$= \frac{-c_3}{5!}$$

$n=4$ $\quad c_6 = \frac{-c_0}{6!}$

$n=5$ $\quad c_7 = \frac{-c_1}{7!}$

$n = 2m$ (even indices)

$$c_{2m} = \frac{(-1)^m c_0}{(2m)!}$$

c_{2m+1} (odd)

$$c_{2m+1} = \frac{(-1)^m c_1}{(2m+1)!}$$

$$y = \sum_{n=0}^{\infty} c_n x^n = \underbrace{\sum_{m=0}^{\infty} \frac{(-1)^m c_0}{(2m)!} x^{2m}}_{\text{even}} + \underbrace{\sum_{m=0}^{\infty} \frac{(-1)^m c_1}{(2m+1)!} x^{2m+1}}_{\text{odd}}$$

Recall:

$$\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} \qquad \sin x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!}$$

$$y = c_0 \cos(x) + c_1 \sin(x)$$